

Large induced forests in planar graphs with girth 4 or 5

François Dross^a, Mickael Montassier^b, and Alexandre Pinlou^c

^a*ENS de Lyon, LIRMM*

^b*Université Montpellier 2, LIRMM*

^c*Université Montpellier 3, LIRMM*

161 rue Ada, 34095 Montpellier Cedex 5, France

francois.dross@ens-lyon.fr, {mickael.montassier, alexandre.pinlou}@lirmm.fr

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Abstract

We give here some new lower bounds on the order of a largest induced forest in planar graphs with girth 4 and 5. In particular we prove that a triangle-free planar graph of order n admits an induced forest of order at least $\frac{6n+7}{11}$, improving the lower bound of Salavatipour [M. R. Salavatipour, Large induced forests in triangle-free planar graphs, *Graphs and Combinatorics*, 22:113–126, 2006]. We also prove that a planar graph of order n and girth at least 5 admits an induced forest of order at least $\frac{44n+50}{69}$.

1 Introduction

Let G be a graph. A *decycling set* or *feedback vertex set* S of G is a subset of the vertices of G such that removing the vertices of S from G yields an acyclic graph. Thus S is a decycling set of G if and only if the graph induced by $V(G) \setminus S$ in G is an induced forest of G . The FEEDBACK VERTEX SET DECISION PROBLEM (which consists of, given a graph G and an integer k , deciding whether there is a decycling set of G of size k) is known to be NP-complete, even restricted to the case of planar graphs, bipartite graphs or perfect graphs [10]. It is thus legitimate to seek bounds for the size of a

decycling set or an induced forest. The smallest size of a decycling set of G is called the *decycling number* of G , and the highest order of an induced forest of G is called the *forest number* of G , denoted respectively by $\phi(G)$ and $a(G)$. Note that the sum of the decycling number and the forest number of G is equal to the order of G (i.e. $|V(G)| = a(G) + \phi(G)$).

Mainly, the community focuses on the following challenging conjecture due to Albertson and Berman [3]:

Conjecture 1 (*Albertson and Berman [3]*). *Every planar graph of order n admits an induced forest of order at least $\frac{n}{2}$.*

Conjecture 1, if true, would be tight (for $n \geq 3$ multiple of 4) because of the disjoint union of the complete graph on four vertices (Akiyama and Watanabe [1] gave examples showing that the conjecture differs from the optimal by at most one half for all n), and would imply that every planar graph has an independent set on at least a quarter of its vertices, the only known proof of which relies on the Four-Color Theorem.

The best known lower bound to date for the forest number of a planar graph is due to Borodin and is a consequence of the acyclic 5-colorability of planar graphs [6]. We recall that an acyclic coloring is a proper vertex coloring such that the graph induced by the vertices of any two color classes is a forest. From this result we obtain the following theorem:

Theorem 2 (*Borodin [6]*). *Every planar graph of order n admits an induced forest of order at least $\frac{2n}{5}$.*

Hosono [9] showed the following theorem as a consequence of the acyclic 3-colorability of outerplanar graphs and showed that the bound is tight.

Theorem 3 (*Hosono [9]*). *Every outerplanar graph of order n admits an induced forest of order at least $\frac{2n}{3}$.*

The tightness of the bound is shown by the example in Figure 1.



Figure 1: Example to prove the tightness of Theorem 3.

Other results were deduced from results on acyclic coloring, for other classes of graphs. Fertin et al. [8] gave such results for several classes of graphs, stated in Table 1.

Family \mathcal{F}	Forest number:	
	Lower bound	Upper bound
Planar	$\frac{2n}{5}$	$\lceil \frac{n}{2} \rceil$
Planar with girth 5, 6	$\frac{n}{2}$	$\frac{7n}{10} + 2$
Planar with girth ≥ 7	$\frac{2n}{3}$	$\frac{5n}{6} + 1$

Table 1: Bounds on the forest number for some families \mathcal{F} of graphs [8].

Akiyama and Watanabe [1], and Albertson and Rhaas [2] independently raised the following conjecture:

Conjecture 4 (*Akiyama and Watanabe [1], and Albertson and Rhaas [2]*). *Every bipartite planar graph of order n admits an induced forest of order at least $\frac{5n}{8}$.*

This conjecture, if true, would be tight for n multiple of 8: for example if G is the disjoint union of k cubes, then we have $a(G) = 5k$ and G has order $8k$ (see Figure 2). Motivated by Conjecture 4, Alon [4] proved the following theorem using probabilistic methods:

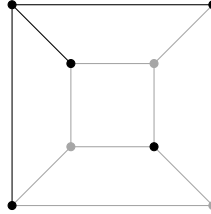


Figure 2: The cube admits an induced forest on five of its vertices, but no induced forest on six or more of its vertices.

Theorem 5 (*Alon [4]*). *There exist some $b > 0$ and $b' > 0$ such that:*

- *For every bipartite graph G with n vertices and average degree at most d (≥ 1), $a(G) \geq (\frac{1}{2} + e^{-bd^2})n$.*
- *For every $d \geq 1$ and all sufficiently large n there exists a bipartite graph with n vertices and average degree at most d such that $a(G) \leq (\frac{1}{2} + e^{-b'\sqrt{d}})n$.*

The lower bound was later improved by Colon et al. [7] to $a(G) \geq (1/2 + e^{-b''d})n$ for a constant b'' .

Conjecture 4 also led to some research for lower bounds of the forest number of triangle-free planar graphs (as a superclass of bipartite planar graphs). Alon et al. [5] proved the following theorems and corollary:

Theorem 6 (Alon et al. [5]). *Every triangle-free graph of order n and size m admits an induced forest of order at least $n - \frac{m}{4}$.*

Corollary 7 (Alon et al. [5]). *Every triangle-free cubic graph of order n admits an induced forest of order at least $\frac{5n}{8}$.*

Theorem 8 (Alon et al. [5]). *Every connected graph with maximum degree Δ , order n , and size m admits an induced forest of order at least $\alpha(G) + \frac{n - \alpha(G)}{(\Delta - 1)^2}$.*

Theorem 6 is tight because of the union of cycles of length 4.

In a planar graph with girth at least g , order n and size m with at least a cycle, the number of faces is at most $2m/g$ (since all the faces' boundaries have length at least g). Then, by Euler's formula, $2m/g \geq m - n + 2$, and thus $m \leq (g/(g - 2))(n - 2)$. In particular, triangle-free planar graphs of order $n \geq 3$ have size at most $2n - 4$.

As a consequence of Theorem 6, for G a triangle-free planar graph of order n , $a(G) \geq n/2$. This lower bound was improved for $n \geq 1$ by Salavatipour [12].

Theorem 9 (Salavatipour [12]). *Every triangle-free planar graph of order n and size m admits an induced forest of order at least $\frac{29n - 6m}{32}$ and thus at least $\frac{17n + 24}{32}$.*

In 2010, Kowalik et al. [11] proposed that for triangle-free planar graphs of order n and size m , $a(G) \geq \frac{119n - 24m - 24}{128} \geq \frac{71n + 72}{128}$. However, it seems that the proof has a flaw. We give here an infinite family of counter-examples for $a(G) \geq \frac{119n - 24m - 24}{128}$ (see Section 2). We propose an improvement of Theorem 9, which thus leads to the best known bound to our knowledge (see Section 2):

Theorem 10. *Every triangle-free planar graph of order n and size m admits an induced forest of order at least $\max\{\frac{38n - 7m}{44}, n - \frac{m}{4}\}$.*

Hence by Euler's formula the following corollary holds:

Corollary 11. *Every triangle-free planar graph of order $n \geq 1$ admits an induced forest of order at least $\frac{6n + 7}{11}$.*

Kowalik et al. [11] made the following conjecture on planar graph of girth at least 5:

Conjecture 12 (*Kowalik et al. [11]*). *Every planar graph with girth at least 5 and order n admits an induced forest of order at least $7n/10$.*

This conjecture, if true, would be tight for n multiple of 20, as shown by the example of the union of dodecahedron, given by Kowalik et al. [11] (see Figure 3). We prove the following theorem which is a first step toward

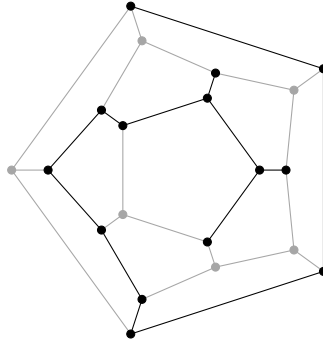


Figure 3: The dodecahedron admits an induced forest on fourteen of its vertices, but no induced forest on fifteen or more of its vertices.

Conjecture 12 (see Section 3):

Theorem 13. *Every planar graph with girth at least 5, order n and size m admits an induced forest of order at least $n - \frac{5m}{23}$.*

Hence by Euler's formula the following corollary holds:

Corollary 14. *Every planar graph with girth at least 5 and order $n \geq 1$ admits an induced forest of order at least $\frac{44n+50}{69}$.*

From Theorem 13 we can deduce, with Euler's formula (which implies that $m \leq (g/(g-2))(n-2)$), the following corollary:

Corollary 15. *Every planar graph with girth at least $g \geq 5$ and order $n \geq 1$ admits an induced forest of order at least $n - \frac{(5n-10)g}{23(g-2)}$.*

Finally, we summarize lower and upper bounds in Table 2. The upper bounds for girth 6 and 7 are obtained by the graphs in Figures 4 and 5. There is no bigger induced forest for any of them since all vertices have degree at most 3, and thus at least one vertex per two faces have to be removed.

Girth higher than	Lower bound for $a(G)$	$a(G)$ for a graph of this class
4	$\frac{6n+7}{11}$	$\frac{5n}{8}$
5	$\frac{44n+50}{69}$	$\frac{7n}{10}$
6	$\frac{31n+30}{46}$	$\frac{23n}{30}$
7	$\frac{16n+14}{23}$	$\frac{17n}{21}$

Table 2: Our lower bounds on $a(G)$ for G planar graph of high enough girth, compared to the best possible lower bounds for $a(G)$ on the corresponding classes of graphs.

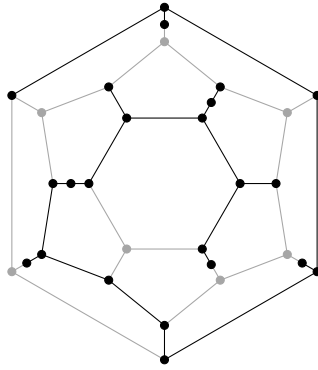


Figure 4: A planar graph of girth 6 on 30 vertices that admits an induced forest on 23 of its vertices, but no induced forest on 24 or more of its vertices.

2 Proof of Theorem 10

We first give a counter-example to the bound of Kowalik et al. [11]: we consider the disjoint union of k cubes. There are $8k$ vertices and $12k$ edges, hence Kowalik et al.'s lower bound tells us that there is an induced forest of size at least $\frac{119(8k)-24(12k)-24}{128} = 5k + (k-1)\frac{3}{16}$. However there cannot be an induced forest of more than 5 vertices in a cube (see Figure 2), and thus the biggest induced forest in our graph contains $5k$ vertices, which contradicts the lower bound. Furthermore, by increasing k , we can see that the biggest induced forest can be arbitrarily smaller than the supposed lower bound.

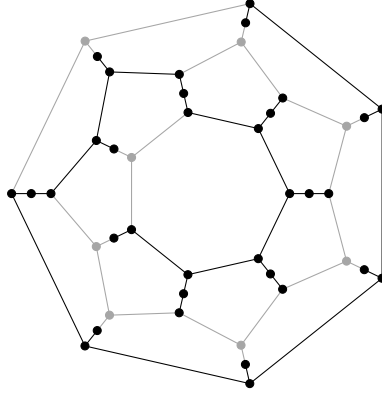


Figure 5: A planar graph of girth 7 on 42 vertices that admits an induced forest on 34 of its vertices, but no induced forest on 35 or more of its vertices.

The proofs of Theorems 10 and 13 follow the same scheme. They consist in looking for a minimal counter-example G , proving some structural properties on G and concluding that it cannot verify Euler's formula, which is contradictory.

Consider $G = (V, E)$. For a set $S \subset V$, let $G - S$ be the graph constructed from G by removing the vertices of S and all the edges incident to some vertex of S . If $x \in V$, then we denote $G - \{x\}$ by $G - x$. For a set S of vertices such that $S \cap V = \emptyset$, let $G + S$ be the graph constructed from G by adding the vertices of S . If $x \notin V$, then we denote $G + \{x\}$ by $G + x$. For a set F of pairs of vertices of G such that $F \cap E = \emptyset$, let $G + F$ be the graph constructed from G by adding the edges of F . If e is a pair of vertices of G and $e \notin E$, we denote $G + \{e\}$ by $G + e$. For a set $W \subset V$, we denote by $G[W]$ the subgraph of G induced by W .

We call a vertex of degree d , at least d and at most d , a d -vertex, a d^+ -vertex and a d^- -vertex respectively. Similarly, we call a cycle of length l , at least l and at most l a l -cycle, a l^+ -cycle and a l^- -cycle respectively, and by extension a face of length l , at least l and at most l a l -face, a l^+ -face and a l^- -face respectively.

Let \mathcal{P}_4 be the class of triangle-free planar graphs, and \mathcal{P}_5 be the class of planar graphs of girth at least 5.

We will prove of the following more general statement than Theorem 10:

Theorem 16. *If a and b are positive constants such that equations (1)–(5)*

are verified, then $a(G) \geq an - bm$ for all $G \in \mathcal{P}_4$.

$$0 \leq a \leq 1 \tag{1}$$

$$0 \leq b \tag{2}$$

$$a - 6b \leq 0 \tag{3}$$

$$3a - 10b \leq 1 \tag{4}$$

$$8a - 12b \leq 5 \tag{5}$$

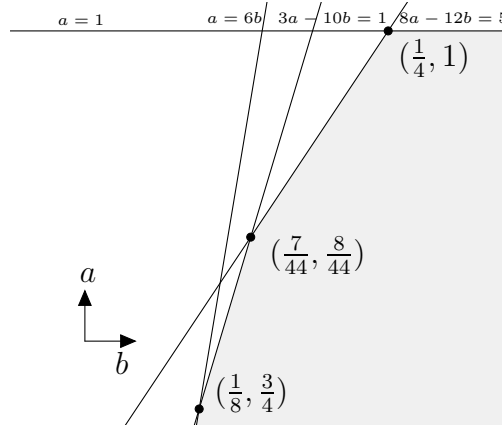


Figure 6: The top-left part of the polygon of the constraints on a and b .

This series of inequalities defines a polygon represented in Figure 6, and for a triangle-free planar graph of given order n and size m , the highest lower bound will be given by maximizing $an - bm$ for a and b in this polygon. This maximum will be achieved at a vertex of the polygon. Moreover, by Euler's formula, every triangle-free planar graph of order $n \geq 3$ and size m satisfies $0 \leq m \leq 2n - 4$. Therefore for $n \geq 3$ the maximum will always be achieved at the intersection of either $3a - 10b = 1$ and $8a - 12b = 5$, or $8a - 12b = 5$ and $a = 1$. The corresponding intersections are $(b, a) = (\frac{7}{44}, \frac{38}{44})$ and $(b, a) = (\frac{1}{4}, 1)$, represented in Figure 6.

Let us show that any of the two lower bounds can be higher than the other, for graphs of arbitrarily high order.

For the disjoint union of k cubes (which is a graph of order $8k$ and size $12k$), the two lower bounds are equal to $5k$.

We consider now a graph composed of k disjoint cubes, where we remove an edge from each cube. This graph has $8k$ vertices and $11k$ edges. In this

case we have $n - \frac{m}{4} = \frac{21}{4}k > \frac{38n-7m}{44} = \frac{227}{44}k$. More simply, for an independent set, $n - \frac{m}{4} = n > \frac{38n-7m}{44} = \frac{38n}{44}$.

We now consider a graph composed of k disjoint cubes, where we add an edge from each cube to the next one and an edge from the last one to the first one. This graph has $8k$ vertices and $13k$ edges. In this case, we have $n - \frac{m}{4} = \frac{19}{4}k < \frac{38n-7m}{44} = \frac{213}{44}k$. For a quadrangulation on n vertices and $2n - 4$ edges (i.e. a planar graph on n vertices that has only 4-faces), $n - \frac{m}{4} = \frac{n}{2} + 1 < \frac{38n-7m}{44} = \frac{6n+7}{11}$.

Let us now proceed to the proof of Theorem 16. For this proof we mainly adapt the methods of Kowalik et al. [11].

Let $G = (V, E)$ be a counter-example to Theorem 16 with the minimum order. Let $n = |V|$ and $m = |E|$. We will use the scheme presented in Observation 17 for most of our lemmas.

Observation 17. *Let α, β, γ be integers satisfying $\alpha \geq 1, \beta \geq 0, \gamma \geq 0$ and $a\alpha - b\beta \leq \gamma$.*

Let $H^ \in \mathcal{P}_4$ be a graph with $|V(H^*)| = n - \alpha$ and $|E(H^*)| \leq m - \beta$.*

By minimality of G , H^ admits an induced forest of order at least $a(n - \alpha) - b(m - \beta)$.*

For all induced forest F^ of H^* of order at least $a(n - \alpha) - b(m - \beta)$, if there is an induced forest F of G of order at least $|V(F^*)| + \gamma$, then we get a contradiction: as $a\alpha - b\beta \leq \gamma$, we have $|V(F)| \geq an - bm$.*

Table 3 contains the values of (α, β, γ) that will be used throughout this section. For each one, the inequality $a\alpha - b\beta \leq \gamma$ is a consequence of the constraints (1)–(5).

We will now prove a series of lemmas on the structure of G .

Lemma 18. *Graph G is 2-edge-connected.*

Proof. By contradiction, suppose $V(G)$ is partitioned into two partite sets V_1 and V_2 such that there is at most one edge between vertices of V_1 and V_2 . Consider graph $G[V_i]$ induced by the vertices of V_i (for $i = 1, 2$) with $n_i = |V_i|$ vertices and $m_i = |E(G[V_i])|$ edges. By minimality of G , $G[V_i]$ admits an induced forest, say F_i , with at least $an_i - bm_i$ vertices. Now the union of F_1 and F_2 (more formally, $G[V(F_1) \cup V(F_2)]$) is an induced forest of G having at least $an_1 - bm_1 + an_2 - bm_2 = a(n_1 + n_2) - b(m_1 + m_2) \geq an - bm$ vertices as $m \geq m_1 + m_2$. A contradiction. \square

In particular, Lemma 18 implies that there is no 1^- -vertex in G .

Lemma 19. *Every vertex in G has degree at most 5.*

α	β	γ	proof
1	6	0	(3)
2	5	1	$((1) + (4))/2$
3	5	2	$(3(1) + (4))/2$
1	1	1	$(1) + (2)$
5	9	3	$((1) + (3) + (5))/2$
6	8	4	$((1) + (5)) * 2/3$
4	10	2	$(1) + (4)$
7	13	4	$((1) + 3(4) + 4(5))/6$
3	10	1	(4)
8	12	5	(5)
6	14	3	$((3) + (4) + (5))/2$
8	19	4	$((1) + (3) + 2(4) + (5))/2$
9	24	4	$((3) + 3(4) + (5))/2$
10	23	5	$((1) + 9(4) + 4(5))/6$
9	19	5	$(3(1) + (3) + 2(4) + (5))/2$

Table 3: The various triples (α, β, γ) and the combinations of inequalities which imply $a\alpha - b\beta \leq \gamma$.

Proof. By contradiction, suppose $v \in V(G)$ is a 6^+ -vertex. Observation 17 applied to $H^* = G - v$ with $(\alpha, \beta, \gamma) = (1, 6, 0)$ and $F = F^*$ completes the proof. \square

Lemma 20. *If v is a 3-vertex adjacent to a 4^+ -vertex w in G , then the two other neighbors of v have a common neighbor different from v .*

Proof. Let x and y be the two neighbors of v different from w . Suppose that they do not have a common neighbor different from v . Let $H^* = G + xy - \{w, v\}$. Graph H^* has $n-2$ vertices and $m' \leq m-5$ edges. As x and y do not have a common neighbor in G other than v , the addition of the edge xy does not create any triangle in H^* , thus $H^* \in \mathcal{P}_4$. Let F' be any induced forest of H^* . Adding v to F' (more formally, consider $G[V(F') \cup \{v\}]$) leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$ completes the proof. \square

Lemma 21. *There is no 2-vertex adjacent to a 4^+ -vertex in G .*

Proof. Let v be a 2-vertex adjacent to a 4^+ -vertex w and $H^* = G - \{v, w\}$. Graph H^* has $n-2$ vertices and $m' \leq m-5$ edges. Let F' be any induced forest of H^* . Adding v to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$ completes the proof. \square

Lemma 22. *There is no 3-vertex adjacent to two 2-vertices in G .*

Proof. Let v be a 3-vertex adjacent to two 2-vertices u and w and $H^* = G - \{u, v, w\}$. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^* . Adding u and w to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ completes the proof. \square

Lemma 23. *Every vertex in G has degree at least 3.*

Proof. Let v be a 2-vertex.

Suppose that v has a neighbor u of degree 2 and a neighbor w of degree 3. Let $H^* = G - \{u, v, w\}$. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^* . Adding u and v to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction.

Suppose that v has two neighbors of degree 3, say u and w . Consider three cases according to the number of neighbors u and w have in common.

- Suppose u and w have only v in common. Let $H^* = G + uw - v$. Graph H^* has $n - 1$ vertices and $m' = m - 1$ edges. Observe that $H^* \in \mathcal{P}_4$. Let F' be any induced forest of H^* . Adding v to F' (more formally, consider $G[V(F') \cup \{v\}]$) does not create any cycle (the edge uw is just subdivided in uv, vw). Observation 17 applied to $(\alpha, \beta, \gamma) = (1, 1, 1)$ leads to a contradiction.
 - Suppose u and w have two neighbors in common, say v and x . Let y be the last neighbor of u . By Lemma 22, both x and y have degree at least 3. Note that x and y are not adjacent because G has girth at least 4. Let $H^* = G - \{u, v, w, x, y\}$. Graph H^* has $n - 5$ vertices and, since y and w are not adjacent (otherwise u and w have three common neighbors), $m' \leq m - 9$ edges. Let F' be any induced forest of H^* . Adding u, v and w to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction.
 - Suppose u and w have three neighbors in common. Let x and y be the ones that are not v . Suppose x is a 4^+ -vertex and let $H^* = G - \{u, v, w, x, y\}$. Graph H^* has $n - 5$ vertices and $m' \leq m - 9$ edges (recall that y is a 3^+ -vertex by Lemma 22). Let F' be any induced forest of H^* . Adding u, v and w to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction.
- W.l.o.g. we assume that x and y are 3-vertices. Let z be the third neighbor of x . Let $H^* = G - \{u, v, w, x, y, z\}$. Graph H^* has $n -$

6 vertices and $m' \leq m - 8$ edges. Let F' be any induced forest of H^* . Adding u, v, x and y to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (6, 8, 4)$ leads to a contradiction.

Therefore, by Lemmas 18 and 21, every 2-vertex has only neighbors of degree 2. As G is connected (Lemma 18), either G does not have any 2-vertex or it is 2-regular. If G is 2-regular, then G is a n -cycle and thus $m = n$. Since $G \in \mathcal{P}_4$, we have $n \geq 4$. It is clear that G has an induced forest of size $n - 1$. Recall that $8a - 12b \leq 5$ and $a \leq 1$; this gives that $4(a - b) \leq 3$. Since $n \geq 4$, we can deduce that $an - bm = (a - b)n \leq n - 1$. This contradicts the fact that G is a counter-example. Therefore, G has minimum degree at least 3. This completes the proof. \square

Lemma 24. *There is no 4-cycle in G with*

- *at least one 4^+ -vertex and two opposite 3-vertices*
- *or one 3-vertex opposite to a 4-vertex that has an edge going to the interior of the cycle and one going to the exterior of it.*

In particular there is no 4-cycle with exactly three 3-vertices in G .

Proof. • Let $C = v_0v_1v_2v_3$ be a cycle such that v_0 and v_2 have degree 3 and v_3 is a 4^+ -vertex. Suppose v_1 is a 4^+ -vertex. Let $H^* = G - C$. Graph H^* has $n - 4$ vertices and $m' \leq m - 10$ edges. Let F' be any induced forest of H^* . Adding v_0 and v_2 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ leads to a contradiction. Therefore v_1 has degree 3.

Let u_0, u_1 and u_2 be the third neighbors of v_0, v_1 , and v_2 , respectively. Suppose $u_0 = u_2$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0\}$. Graph H^* has $n - 5$ vertices and $m' \leq m - 9$ edges. Let F' be any induced forest of H^* . Adding v_0, v_1 and v_2 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction. So u_0 and u_2 are distinct.

By Lemma 20, $u_0u_1 \in E$ and $u_1u_2 \in E$. Assume u_0 (or u_2) has at most one neighbor $w \notin \{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 13$ edges. Let F' be any induced forest of H^* . Adding v_0, v_1, v_2 and u_0 to H^* leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (7, 13, 4)$ leads to a contradiction. Thus both of the vertices u_0 and u_2 have at least two neighbors that are not in $\{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_2\}$. Graph

H^* has $n - 6$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^* . Adding the vertices v_0, v_1 and v_2 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction.

- Let $C = v_0v_1v_2v_3$ be a cycle such that v_0 is a 3-vertex and v_2 is a 4-vertex with an edge going to the interior of the cycle and one going to the exterior of it. If v_1 and v_3 have degree 3, then we fall into the previous case. Therefore w.l.o.g. v_1 is a 4^+ -vertex. Let $H^* = G - C$. Graph H^* has $n - 4$ vertices and $m' \leq m - 10$ edges. Let F' be any induced forest of H^* . Adding v_0 and v_2 to F' leads to an induced forest of G . Indeed, if adding v_2 creates a cycle, then there is a path from the interior to the exterior of C in H^* , which is impossible. Observation 17 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ completes the proof. \square

Lemma 25. *There is no 4-face with four 3-vertices in G .*

Proof. Suppose that there is such a 4-face $C = v_0v_1v_2v_3$, and let u_i be the third neighbor of v_i for $i = 0..3$. In the following, we consider the indices of the u_i and v_i modulo 4. If for some $i_0 \in \{0, 1, 2, 3\}$, $u_{i_0} = u_{i_0+1}$, then we have a triangle. Suppose now that $u_{i_0} = u_{i_0+2}$ for some $i_0 \in \{0, 1, 2, 3\}$, w.l.o.g. say $i_0 = 0$. In the cycle $v_0v_1v_2u_0$, the vertices v_0 and v_2 are two opposite 3-vertices. By Lemma 24, u_0 is a 3-vertex. Observe that u_1v_1 and u_3v_3 are separated by the cycle $v_0v_1v_2u_0$. Hence one of them is a bridge, contradicting Lemma 18.

Therefore all the u_i are distinct. We now consider the question of the presence or not of the edges u_iu_{i+1} . Consider the case $u_iu_{i+1} \notin E$ and $u_{i+1}u_{i+2} \notin E$ for some $i \in \{0, 1, 2, 3\}$, w.l.o.g. say $i = 0$. If $u_0u_2 \in E$, then either $u_2u_3 \notin E$ or $u_0u_3 \notin E$ (otherwise G has a triangle), and $u_1u_3 \notin E$ by planarity of G . Therefore up to the permutation of the indices, $u_0u_1 \notin E$, $u_1u_2 \notin E$ and $u_0u_2 \notin E$. We then define $H^* = G + x + \{xu_0, xu_1, xu_2\} - \{v_0, v_1, v_2, v_3\}$. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges and belongs to \mathcal{P}_4 as u_0u_1 , u_0u_2 and u_1u_2 are not in E . Let F' be any induced forest of H^* . Let F be the subgraph of G induced by $V(F') \setminus \{x\}$ plus v_0 , v_1 and v_2 if $x \in F'$ or plus v_0 and v_2 if $x \notin F'$. Subgraph F is an induced forest of G . Hence, Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction. Therefore there must be an i such that $u_iu_{i+1} \in E$ and $u_{i+2}u_{i+3} \in E$, w.l.o.g. $u_0u_1 \in E$ and $u_2u_3 \in E$.

Let $G' = G - C$. Graph G' has $n - 4$ vertices and $m - 8$ edges.

Let us now count, for each of the u_i 's, the number of the neighbors of u_i that are not in $A = \{v_0, v_1, v_2, v_3, u_0, u_1, u_2, u_3\}$. The edges that are known

in $G[A]$ are represented in Figure 7.

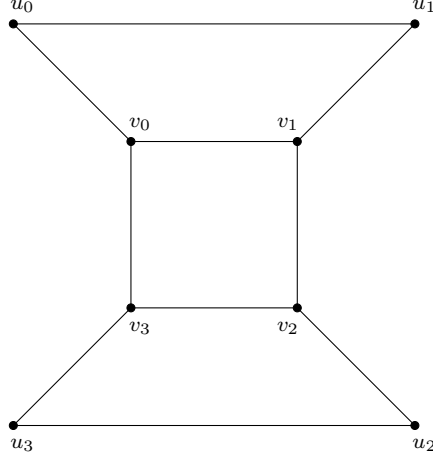


Figure 7: The graph $G[A]$ (only the edges that are known to be there are represented).

- Suppose w.l.o.g. u_0 has only neighbors in A , and another $u_{i'}$ has at most one neighbor not in A . Let $H^* = G' - \{u_0, u_1, u_2, u_3\}$. Graph H^* has $n - 8$ vertices. By Lemma 23, each of the u_i has degree at least 3. Graph H^* has $m' \leq m - 12$ edges. Let F' be any induced forest of H^* . Adding the vertices $u_0, u_{i'}, v_1, v_2$ and v_3 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (8, 12, 5)$ leads to a contradiction.
- Suppose w.l.o.g. u_0 has at most one neighbor not in A , and all the other u_i have each at least one neighbor not in A . Vertex u_0 is not adjacent both to u_2 and u_3 since G has girth at least 4. Let i_0 be such that $i_0 \neq 0$ and $u_0 u_{i_0} \notin E$ (either $i_0 = 2$ or $i_0 = 3$). Let $H^* = G' - \{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$ (we remove all the vertices of A except u_{i_0}). Graph H^* has $n - 7$ vertices. Let us count the number of edges in G' that have an endvertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. If $i_0 = 2$, then there are at least two edges for the neighbors of u_1 and u_3 that are not in A , plus the edges $u_0 u_1$ and $u_2 u_3$, plus one edge since u_0 has degree at least 3, thus at least 5 edges of H^* have an endvertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. If $i_0 = 3$, then there are at least two edges for the neighbors of u_1 and u_2 that are not in A , plus the edges $u_0 u_1$ and $u_2 u_3$, plus one edge since u_0 has degree at least 3, thus at least 5 edges of H^* have an

endvertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. In both cases, H^* has $m' \leq m - 13$ edges. Let F' be any induced forest of H^* . Adding the vertices u_0, v_1, v_2 and v_3 to F' leads to an induced forest of G , since there is no path between u_0 and u_{i_0} in $G[\{v_1, v_2, v_3, u_0, u_{i_0}\}]$. Observation 17 applied to $(\alpha, \beta, \gamma) = (7, 13, 4)$ leads to a contradiction.

- So all the u_i have at least two neighbors not in A . Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_2\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges, and if F' is any induced forest in H^* , then adding the vertices v_0, v_1 and v_2 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction and completes the proof.

□

Lemma 26. *There is no separating 4-cycle with four 3-vertices in G .*

Proof. Let $C = v_0v_1v_2v_3$ be such a cycle. We will consider the indices of the v_i modulo 4 in what follows. Since G is 2-edge-connected (Lemma 18), two of the v_i have their third neighbor in the interior of C , and the two other have theirs outside of it. There is a v_i such that the third neighbors of v_{i+1} and v_{i+2} are separated by C , w.l.o.g. for $i = 0$. Then let u be the third neighbor of v_0 . Let $H^* = G - C - u$. Graph H^* has $n - 5$ vertices, and $m' \leq m - 9$ edges. Let F' be any induced forest of H^* . Adding the vertices v_0, v_1 and v_2 to F' leads to a forest of G , thus Observation 17 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$ leads to a contradiction. □

Lemma 27. *There is no 3-vertex adjacent to a 5-vertex in G .*

Proof. Let v be a 3-vertex adjacent to a 5-vertex u . Let w and x be the two other neighbors of v .

We first assume that w or x , w without loss of generality, is a 4^+ -vertex. Let $H^* = G - \{u, v, w\}$. Graph H^* has $n - 3$ vertices and $m' \leq m - 10$ edges. Let F' be any induced forest of H^* . Adding v to F' leads to an induced forest of G . Thus Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$ leads to a contradiction.

Therefore w and x are 3-vertices. By Lemma 20, w and x have a common neighbor (distinct from v), which has degree 3 by Lemma 24. Finally Lemmas 25 and 26 lead to a contradiction, completing the proof. □

Lemma 28. *There is no separating 4-cycle with at least two 3-vertices in G .*

Proof. Let $C = v_0v_1v_2v_3$ be such a cycle. By Lemmas 24 and 26, C has exactly two 3-vertices. By Lemmas 23, 24 and 27, the two 3-vertices are adjacent, the two other vertices have degree 4 and none of the 4-vertices has a neighbor inside C and the other one outside C . W.l.o.g. the 3-vertices are v_0 and v_1 . Let u_0 and u_1 be the third neighbors of v_0 and v_1 respectively.

If $u_0v_2 \in E$ or $u_1v_3 \in E$, say $u_0v_2 \in E$ w.l.o.g., then either $v_0v_1v_2u_0$ or $v_0v_3v_2u_0$ has a 3-vertex (v_0) opposite to a 4-vertex (v_2) with an edge going inside and one going outside of it, contradicting Lemma 24. Therefore $u_0v_2 \notin E$ and $u_1v_3 \notin E$.

By Lemma 20, $u_0u_1 \in E$; thus C does not separate u_0 and u_1 , say u_0 and u_1 are in the exterior of C up to changing the plane embedding. By Lemmas 23–27, u_0 and u_1 are 4-vertices. At least one of v_2 or v_3 , say v_2 , has two neighbors inside of C (otherwise the cycle is not separating). Let $H^* = G - \{v_0, v_1, v_3, u_1\}$. Graph H^* has $n - 4$ vertices and $m' \leq m - 10$ edges, and if F' is any induced forest of H^* , then adding v_0 and v_1 to F' leads to an induced forest of G (since v_2 is only connected to the interior and u_0 to the exterior of C). Observation 17 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ completes the proof. \square

Lemma 29. *There is no 4-face with exactly two 3-vertices in G .*

Proof. Let $C = v_0v_1v_2v_3$ be such a face. By Lemmas 23 and 24 the two 3-vertices are adjacent. W.l.o.g. v_0 and v_1 have degree 3, and v_2 and v_3 have degree 4 (by Lemmas 23 and 27). Let u_0 and u_1 be the third neighbors of v_0 and v_1 respectively. By Lemma 20 applied to v_0 and v_3 , and v_1 and v_2 , $u_0u_1 \in E$. Then by Lemma 28, $v_0v_1u_1u_0$ cannot be a separating cycle, and so it is the boundary of some 4-face. If both u_0 and u_1 have degree 3, we have a contradiction by Lemma 25. If one has degree 3 and the other has degree at least 4, we have a contradiction by Lemma 24. Finally, by Lemma 27, u_0 and u_1 are 4-vertices.

If v_2 is adjacent to u_0 , then $u_0v_0v_1v_2$ is a separating 4-cycle, with two 3-vertices, contradicting Lemma 28. Hence v_2u_0 is not in E . Similarly, v_3u_1 is not in E . Since $G \in \mathcal{P}_4$, either u_0 and v_2 do not have a common neighbor, or u_1 and v_3 do not have a common neighbor. By symmetry assume that u_0 and v_2 do not have a common neighbor. Let $H^* = G + u_0v_2 - \{u_1, v_0, v_1, v_3\}$. Graph H^* has $n - 4$ vertices, $m' \leq m - 10$ edges and belongs to \mathcal{P}_4 . Let F' be any induced forest of H^* . Adding v_0 and v_1 to F' leads to an induced forest of G (intuitively the edge u_0v_2 is just subdivided). Observation 17 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$ completes the proof. \square

Lemma 30. *There is no 4-cycle with at least two 3-vertices in G .*

Proof. It follows from Lemmas 24, 25, 28 and 29. \square

Lemma 31. *There is no 4-face with exactly one 3-vertex in G .*

Proof. Let $C = v_0v_1v_2v_3$ be such a face. W.l.o.g. v_0 is the 3-vertex and v_1, v_2 and v_3 are 4^+ -vertices. By Lemma 27, v_1 and v_3 are 4-vertices. Let u_0 be the third neighbor of v_0 . Vertex u_0 is different from v_2 and non-adjacent to v_1 and v_3 (G is triangle-free).

Let us first assume that $u_0v_2 \in E$. By Lemmas 23, 27 and 30, u_0 is a 4-vertex. Assume v_2 has degree 5. Let $H^* = G - \{u_0, v_0, v_2\}$. Graph H^* has $n - 3$ vertices and $m - 10$ edges. Let F' be any induced forest of H^* . Adding the vertex v_0 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$ leads to a contradiction. Hence v_2 has degree 4. Then either $v_0v_1v_2u_0$ or $v_0v_3v_2u_0$ has a 3-vertex opposite to a 4-vertex with a neighbor in the interior and one in the exterior of it, contradicting Lemma 24.

Thus u_0 is non-adjacent to v_2 . By Lemma 20, v_1 and u_0 have a common neighbor other than v_0 , say u_1 . It is distinct from all the vertices we defined previously. By Lemma 30 applied to $v_0v_1u_1u_0$, u_0 and u_1 have degree at least 4. By Lemma 27, u_0 has degree exactly 4.

Suppose $u_1v_3 \in E$. As C is a face, the last neighbor of v_1 ($\neq v_0, v_2, u_1$), say w_1 , is not in the interior of C . The cycle $v_0v_1u_1v_3$ separates u_0 and v_2 . Suppose first that $v_0v_1u_1v_3$ does not separate u_0 and w_1 . Then $v_0v_1u_1u_0$ separates v_3 and w_1 . Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1\}$. Graph G^* has $n - 6$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^* . Adding the vertices v_0, v_1 and v_3 to F' leads to an induced forest of G . Hence Observation 17 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction. Therefore $v_0v_1u_1v_3$ separates u_0 and w_1 . Assume u_1 has degree 5. Let $H^* = G - \{u_1, v_0, v_3\}$. Graph H^* has $n - 3$ vertices and $m - 10$ edges. Let F' be any induced forest of H^* . Adding the vertex v_0 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$ leads to a contradiction. Hence u_1 has degree 4. Then $v_0v_1u_1v_3, v_0u_0u_1v_3$ or $v_0v_1u_1u_0$ has a 3-vertex opposite to a 4-vertex with a neighbor in the interior and one in the exterior of it, contradicting Lemma 24.

So u_1 cannot be adjacent to v_3 . As $u_1v_3 \notin E$ and $u_0v_2 \notin E$, by Lemma 20 v_3 and u_0 have a common neighbor distinct from v_0 , say u_3 . By what precedes and by symmetry, it is of degree at least 4 and non-adjacent to v_0, v_1, v_2 and u_1 (it has a role similar to that of u_1 , and is non-adjacent to u_1 because of the girth assumption). See Figure 8 for a reminder of the structure of $G[\{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}]$. Vertex v_0 has degree 3, v_1, v_3 and u_0 are 4-vertices, and v_2, u_1 and u_3 are 4^+ -vertices. Recall that $u_1v_3 \notin E, u_3v_1 \notin E$ and $u_0v_2 \notin E$.

Let w_0, w_1 and w_3 be the fourth neighbors of u_0, v_1 and v_3 respec-

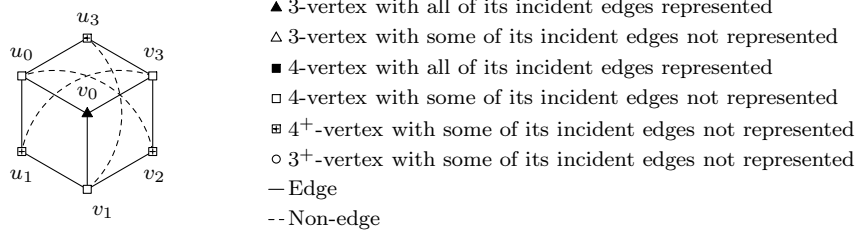


Figure 8: Graph $G[\{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}]$.

tively. In the following we will no longer use the fact that C is a face. By the girth assumption, w_0 is not adjacent to u_1 or u_3 . Suppose w_0 is adjacent to v_1 or to v_3 , say $w_0v_1 \in E$. Then by the girth assumption, $w_0v_2 \notin E$. By Lemma 30 applied to $v_0v_1w_0u_0$, w_0 is a 4^+ -vertex. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0\}$. Graph H^* has $n - 8$ vertices and $m' \leq m - 19$ edges. Let F' be any induced forest of H^* . Adding the vertices v_0, v_1, v_3 and u_0 to F' leads to an induced forest of G . Hence Observation 17 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ leads to a contradiction. So w_0 is not adjacent to v_1 or v_3 . By symmetry, w_0, w_1 and w_3 are distinct.

Suppose $w_0v_2 \in E$. Assume that C separates w_1 and w_3 , or that it does not separate w_1 and w_3 nor w_0 and w_1 . Then either C or $v_0v_1v_2w_0u_0$ separates w_1 and w_3 . Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0\}$. Graph H^* has $n - 8$ vertices and $m' \leq m - 19$ edges. Let F' be any induced forest of H^* . Adding the vertices v_0, v_1, v_3 and u_0 to F' leads to an induced forest of G . Hence Observation 17 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ leads to a contradiction. Thus C does not separate w_1 and w_3 but separates w_1 and w_0 . Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_3\}$. Graph H^* has $n - 8$ vertices and $m' \leq m - 19$ edges. Let F' be any induced forest of H^* . Adding the vertices v_0, v_1, v_3 and u_0 to F' leads to an induced forest of G . Hence Observation 17 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ leads to a contradiction. So $w_0v_2 \notin E$, and similarly $w_1u_3 \notin E$ and $w_3u_1 \notin E$.

Thus the only edges that may or may not exist between the vertices we defined are w_0w_1 , w_0w_3 and w_1w_3 . See Figure 9 for a reminder of the edges and vertices we know to this point. Vertex v_0 has degree 3, v_1, v_3 and u_0 are 4-vertices and v_2, u_1 and u_3 are 4^+ -vertices. Vertices v_0, v_1, v_3 and u_0 have all their incident edges represented in Figure 9.

Suppose $w_0w_1 \notin E$, $w_0w_3 \notin E$, and $w_1w_3 \notin E$. Let $H^* = G + x + \{xw_0, xw_1, xw_3\} - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges, and is in \mathcal{P}_4 . Let F' be any induced forest of H^* . Either $x \in F'$, then the graph induced by $V(F') \cup \{v_0, v_1, v_3, u_0\} \setminus \{x\}$ in G is a

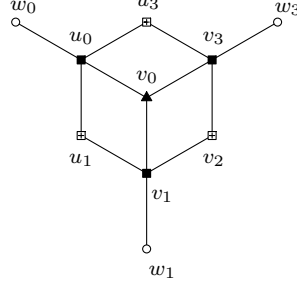


Figure 9: Vertices $v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1$ and w_3 .

forest, or $x \notin F'$, then adding v_1, v_3 and u_0 to F' leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction. Thus there is at least one edge among w_0w_1, w_0w_3 and w_1w_3 . Moreover, since there is no triangle in G , there are no more than two of these edges. W.l.o.g. let us assume that $w_0w_1 \notin E$ and $w_0w_3 \in E$.

Let us now prove some claims that we will use later :

- (a) Suppose that w_0 and w_1 are 4^+ -vertices, or that one is a 3-vertex, the other a 4^+ -vertex, and v_2, u_1 or u_3 has degree 5. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1\}$. Graph H^* has $n - 9$ vertices and $m' \leq m - 24$ edges, and adding v_0, v_1, v_3 and u_0 to any induced forest of H^* leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (9, 24, 4)$ leads to a contradiction.
- (b) Suppose w_0 or w_3 , say w_{i_0} , is a 3-vertex and either one of the w_i is a 4^+ -vertex, or $w_1w_3 \notin E$. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, w_3\}$. Graph H^* has $n - 10$ vertices and $m' \leq m - 23$ edges, and adding v_0, v_1, v_3, u_0 and w_{i_0} to any induced forest of H^* leads to an induced forest of G . Observation 17 applied to $(\alpha, \beta, \gamma) = (10, 23, 5)$ leads to a contradiction.
- (c) Suppose w_0 and w_3 are 3-vertices and w_1 and w_3 are adjacent. Let $H^* = G - \{v_0, v_1, v_3, u_0, u_1, u_3, w_0, w_1, w_3\}$. Graph H^* has $n - 9$ vertices and $m' \leq m - 19$ edges, and adding v_0, v_1, u_0, w_0 and w_3 to any induced forest of H^* leads to an induced forest of G (by planarity, since $w_1w_3 \in E$ and $w_0w_3 \in E$, the cycle $v_0v_1w_1w_3v_3$ separates v_2 from w_0 in G). Observation 17 applied to $(\alpha, \beta, \gamma) = (9, 19, 5)$ leads to a contradiction.

If $w_1w_3 \in E$, then both w_0 and w_3 are 4^+ -vertices (by (b) and (c)), and

by symmetry w_1 is also a 4^+ -vertex, which is impossible (by (a)). Hence $w_1w_3 \notin E$.

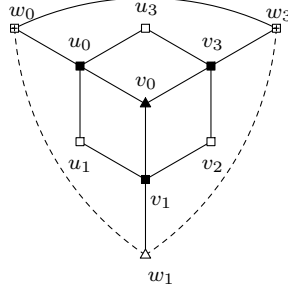


Figure 10: Vertices $v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1$ and w_3 .

Therefore w_0 and w_3 are 4^+ -vertices (by (b)), thus w_1 has degree 3 (by (a)), and v_2, u_1 and u_3 have degree 4 (by (a)) (see Figure 10). Let y_0 and y_1 the two neighbors of w_1 other than v_1 . By Lemma 20 they have a common neighbor other than w_1 , say t . So by Lemmas 27 and 30 in $w_1y_0ty_1$, y_0 and y_1 have degree 4, and by Lemma 20 each one is adjacent either to v_2 or to u_1 . If they are both adjacent to the same one, say v_2 w.l.o.g., then either $v_2v_1w_1y_0$ or $v_2v_1w_1y_1$ is a 4-cycle with a 3-vertex (w_1) opposite to a 4-vertex (v_2) that has both an edge going outside and one going inside of it, which is impossible by Lemma 24. W.l.o.g., say y_0 is adjacent to v_2 and y_1 is adjacent to u_1 . At this point we know that $v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0$ and y_1 are distinct and do not share an edge that we do not already know. See Figure 11 for a reminder of the edges and vertices we know to this point.

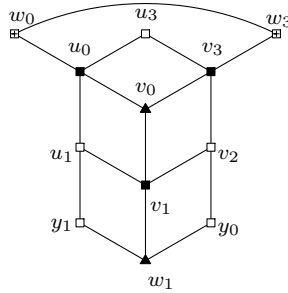


Figure 11: Vertices $v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, w_3, y_0$ and y_1 .

Let z be the neighbor of v_2 different from v_1, v_3 and y_0 . The only edges that may or not be among $v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, y_1$ and z are zy_1

and zu_1 , and as G is triangle-free, there is at most one of those edges. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, y_1, z\}$. Graph H^* has $n - 10$ vertices and $m' \leq m - 23$ edges (recall that u_1 cannot be adjacent both to y_0 and y_1 , and thus is not adjacent to y_0). Adding to any induced forest of H^* the vertices v_0, v_1, v_2, u_1 and w_1 leads to an induced forest of G , so Observation 17 applied to $(\alpha, \beta, \gamma) = (10, 23, 5)$ leads to a contradiction, completing the proof. \square

Lemma 32. *There is no 5-face with only 3-vertices in G .*

Proof. Let $C = v_0v_1v_2v_3v_4$ be such a face, and u_0, u_1, u_2, u_3 , and u_4 be the third neighbors of v_0, v_1, v_2, v_3 , and v_4 respectively. The u_i are all distinct due to the girth assumption and Lemma 28. We will consider the indices of the u_i and v_i modulo 5. There is no edge u_iu_{i+1} for any i due to Lemma 30. Let $H^* = G + x + y + \{xu_0, xu_1, yu_2, yu_3, xy\} - C$. Graph H^* has $n - 3$ vertices and $m - 5$ edges. Let F' be any induced forest of H^* . Let F be the subgraph of G induced by the vertices of $V(F') \setminus \{x, y\}$, plus the vertices v_0 and v_3 , plus v_1 if $x \in V(F')$, and plus v_2 if $y \in V(F')$. Subgraph F is an induced forest of G . Thus Observation 17 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction completing the proof. \square

Lemma 33. *There is no 3-vertex adjacent to a 3-vertex and to a 4-vertex in G .*

Proof. Let v be a 3-vertex adjacent to a 3-vertex u and to a 4-vertex w . Let x be the third neighbor of v . By Lemma 20, x and u have a common neighbor distinct from v which contradicts Lemma 30. \square

For every face f of G , let $l(f)$ be the length of f , and let $c_{4^+}(f)$ be the number of 4^+ -vertices in f . For every vertex v , let $d(v)$ be the degree of v . Let k be the number of faces of G , and for every $3 \leq d \leq 5$ and every $4 \leq l$, let k_l be the number of faces of length l and n_d the number of d -vertices in G .

Each 4-vertex is in the boundary of at most four faces, and each 5-vertex is in the boundary of at most five faces. Therefore the sum of the $c_{4^+}(f)$ over all the 4-faces and 5-faces is $\sum_{f, 4 \leq l(f) \leq 5} c_{4^+}(f) \leq 4n_4 + 5n_5$. From Lemmas 27, 32 and 33 we can deduce that for each 5-face f we have $c_{4^+}(f) \geq 2$. Moreover, by Lemmas 30 and 31, for each 4-face f , $c_{4^+}(f) \geq 4$. Thus $\sum_{f, l(f)=4} c_{4^+}(f) + \sum_{f, l(f)=5} c_{4^+}(f) \geq 4k_4 + 2k_5$. Thus we have the following:

$$4n_4 + 5n_5 \geq 4k_4 + 2k_5$$

By Euler's formula, we have:

$$\begin{aligned}
-12 &= 6m - 6n - 6k \\
&= 2 \sum_{v \in V(G)} d(v) + \sum_{f \in F(G)} l(f) - 6n - 6k \\
&= \sum_{d \geq 3} (2d - 6)n_d + \sum_{l \geq 4} (l - 6)k_l \\
&\geq 2n_4 + 4n_5 - 2k_4 - k_5 \\
&\geq 0
\end{aligned}$$

This is a contradiction, which ends the proof of Theorem 16.

3 Proof of Theorem 13

The proof of Theorem 13 follows the same scheme as that of Theorem 10. We will prove the following more general statement than Theorem 13:

Theorem 34. *If a and b are positive constants such that equations (6)–(9) are verified, then $a(G) \geq an - bm$ for all $G \in \mathcal{P}_5$.*

$$0 \leq a \leq 1 \tag{6}$$

$$0 \leq b \tag{7}$$

$$a - 5b \leq 0 \tag{8}$$

$$11a - 23b \leq 6 \tag{9}$$

This series of inequalities defines a polygon represented in Figure 12, and for a graph in \mathcal{P}_5 of given order n and size m , the highest lower bound will be given by maximizing $an - bm$ for a and b in this polygon. This maximum will be achieved at a vertex of the polygon. Moreover, by Euler's formula, every planar graph of girth at least 5, order $n \geq 4$ and size m satisfies $0 \leq m \leq \frac{5n-10}{3}$. Then for $n \geq 4$ the maximum will always be achieved at the intersection of $11a - 23b = 6$ and $a = 1$. The corresponding intersection is $(b, a) = (\frac{5}{23}, 1)$, represented in Figure 12.

Let $G = (V, E)$ be a counter-example to Theorem 34 of minimum order. Let $n = |V|$ and $m = |E|$. We will use the scheme presented in Observation 35 for most of our lemmas.

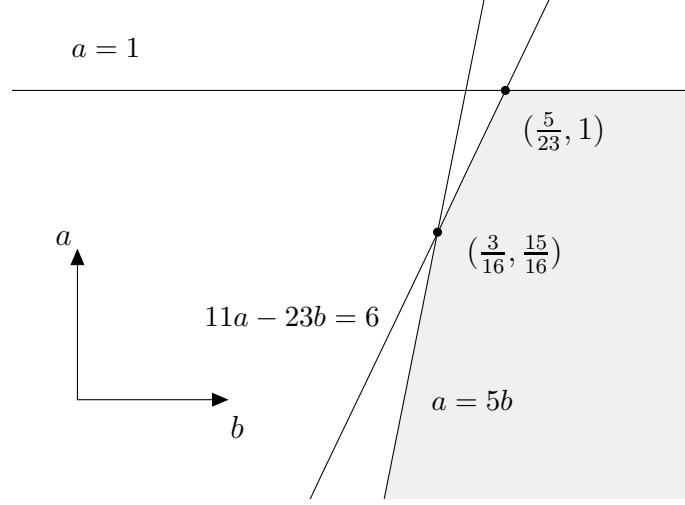


Figure 12: The top-left part of the polygon of the constraints on a and b .

Observation 35. Let α, β, γ be integers satisfying $\alpha \geq 1, \beta \geq 0, \gamma \geq 0$ and $a\alpha - b\beta \leq \gamma$.

Let $H^* \in \mathcal{P}_5$ be a graph with $|V(H^*)| = n - \alpha$ and $|E(H^*)| \leq m - \beta$.

By minimality of G , H^* admits an induced forest of order at least $a(n - \alpha) - b(m - \beta)$.

For all induced forest F^* of H^* of order at least $a(n - \alpha) - b(m - \beta)$, if there is an induced forest F of G of order at least $|V(F^*)| + \gamma$, then we get a contradiction: as $a\alpha - b\beta \leq \gamma$, we have $|V(F)| \geq an - bm$.

Table 4 contains the values of (α, β, γ) that will be used throughout this section. For each one, the inequality $a\alpha - b\beta \leq \gamma$ is a consequence of the constraints (6)–(9).

We will now prove a series of lemmas on the structure of G .

Lemma 36. Graph G is 2-edge-connected.

Proof. See the proof of Lemma 18. □

Lemma 37. Every vertex in G has degree at most 4.

Proof. By contradiction, suppose $v \in V(G)$ has degree at least 5. Observation 35 applied to $H^* = G - v$, $(\alpha, \beta, \gamma) = (1, 5, 0)$ and $F = F'$ leads to a contradiction. □

α	β	γ	proof
1	5	0	(8)
2	5	1	(6) + (8)
3	5	2	2(6) + (8)
5	10	3	3(6) + 2(8)
1	0	1	(6)
6	14	3	((8) + (9))/2
6	10	4	4(6) + 2(8)
7	14	4	(6) + ((8) + (9))/2
7	10	5	5(6) + 2(8)
10	15	7	7(6) + 3(8)
8	14	5	2(6) + ((8) + (9))/2
10	20	6	6(6) + 4(8)
11	19	7	4(6) + (3(8) + (9))/2
12	23	7	(6) + (9)
8	19	4	2(6) + (3(8) + (9))/2
9	15	6	6(6) + 3(8)
11	23	6	(9)
13	23	8	2(6) + (9)

Table 4: The various triples (α, β, γ) and the combinations of inequalities which imply $a\alpha - b\beta \leq \gamma$.

Lemma 38. *If v is a 3-vertex adjacent to a 4-vertex w in G , and if x and y are the two other neighbors of v , then there are two other vertices x' and y' such that $vxx'y'y$ is a cycle.*

Proof. Suppose that there is no cycle as in the statement of the lemma. Let $H^* = G + xy - \{w, v\}$. Graph H^* has $n - 2$ vertices and $m' \leq m - 5$ edges. As there are no x' and y' as in the lemma, adding the edge xy does not create any 4^- -cycle in H^* , and thus $H^* \in \mathcal{P}_5$. Let F' be any induced forest of H^* . Adding v to F' leads to a forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$ completes the proof. \square

Lemma 39. *There is no 2-vertex adjacent to a 4-vertex in G .*

Proof. Let v be a 2-vertex and w a 4-vertex adjacent to v . Let $H^* = G - \{v, w\}$. Graph H^* has $n - 2$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^* . Adding v to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$ completes the proof. \square

Lemma 40. *There is no 3-vertex adjacent to two 2-vertices in G .*

Proof. Let v be a 3-vertex adjacent to two 2-vertices u and w . Let $H^* = G - \{u, v, w\}$. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^* . Adding u and w to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ completes the proof. \square

Lemma 41. *There is no separating 5-cycles with only 3-vertices in G .*

Proof. Let $C = v_0v_1v_2v_3v_4$ be such a cycle. W.l.o.g. v_0 has his third neighbor in the interior of C and v_1 in the exterior of it. Let $H^* = G - C$. Graph H^* has $n - 5$ vertices and $m' = m - 10$ edges. Adding v_0, v_1 and v_3 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (5, 10, 3)$ leads to a contradiction. \square

Lemma 42. *Every vertex in G has degree at least 3.*

Proof. Let v be a 2-vertex in G .

Suppose that v is adjacent to a 2-vertex u and a 3-vertex w . Let $H^* = G - \{u, v, w\}$. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^* . Adding u and v to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction.

Suppose that v is adjacent to two 3-vertices u and w . Consider two cases according to the presence or not of 5-cycles containing uvw .

- Suppose there is no 5-cycle containing uvw . Let $H^* = G + uw - v$. Graph H^* has $n - 1$ vertices and $m - 1$ edges. As there is no 5-cycle containing uvw , adding the edge uw does not create any cycle of length 3 or 4 in H^* , thus $H^* \in \mathcal{P}_5$. Let F' be any induced forest of H^* . Adding v to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (1, 0, 1)$ leads to a contradiction.
- Suppose there is a 5-cycle containing uvw , say $uvwxy$. By Lemma 40, both x and y are 3^+ -vertices.

Suppose x or y , say x , has degree 3, and the other one has degree 4. Let $H^* = G - \{u, v, w, x, y\}$. Graph H^* has $n - 5$ vertices and, since there is no chord in the 5-cycle, $m' = m - 10$ edges. Let F' be any induced forest of H^* . Adding u, v and x to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (5, 10, 3)$ leads to a contradiction.

Suppose both x and y have degree 3. Let u', w', x' , and y' be the third neighbors of u, w, x and y respectively. They are all distinct by the girth assumption. By Lemma 40, u' and w' are 3^+ -vertices. Suppose x' or y' , say x' , has degree 2. Let $H^* = G - \{u, v, w, x, y, x'\}$. Graph H^*

has $n - 6$ vertices and $m' = m - 10$ edges. Adding u, v, x and x' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (6, 10, 4)$ leads to a contradiction.

Hence u', w', x' and y' are 3^+ -vertices. Suppose u' or y' is a 4-vertex. By the girth assumption, $u'y' \notin E$. Let $H^* = G - \{u, v, w, x, y, u', y'\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 14$ edges. Adding u, v, w and y to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 14, 4)$ leads to a contradiction. Therefore u', w', x' and y' are 3-vertices.

Let us now show that $u'x' \notin E$ (and by symmetry $w'y' \notin E$). Suppose by contradiction that $u'x' \in E$. By Lemma 41, the cycle $uyxx'u'$ bounds a face, hence the cycle $uvwxx'u'$ separates y' from the third neighbor of x' . Let $H^* = H - \{u, v, w, x, y, u', x'\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 10$ edges. Adding u, v, x, y and x' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 10, 5)$ leads to a contradiction.

Suppose that there is no vertex adjacent to both u' and y' . Let $H^* = G - \{u, v, w\} + u'y$. Graph H^* has $n - 3$ vertices and $n - 5$ edges, and has girth at least 5 since $u'x' \notin E$ and there is no vertex adjacent to u' and y' . Adding u and v to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction. Hence there is a vertex z adjacent to u' and y' .

Suppose that there is no vertex adjacent to x' and y' . Let $H^* = G - \{v, w, x\} + x'y$. Graph H^* has $n - 3$ vertices and $n - 5$ edges, and has girth at least 5 since $u'x' \notin E$ and there is no vertex adjacent to x' and y' . Adding x and v to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ leads to a contradiction. Hence there is a vertex z' adjacent to x' and y' .

Suppose z is a 2-vertex. Vertices z and z' are distinct, and non-adjacent. Let $H^* = G - \{u, v, w, x, y, u', x', y', z, z'\}$. Graph H^* has $n - 10$ vertices and $n - 15$ edges. Adding u, v, x, u', x', y' and z to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (10, 15, 7)$ leads to a contradiction.

Therefore z is a 3^+ -vertex. Let $H^* = G - \{u, v, w, x, y, u', y', z\}$. Graph H^* has $n - 8$ vertices and $n - 14$ edges ($w' \neq z$, since $w'y' \notin E$). Adding u, v, x, u' , and y' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (8, 14, 5)$ leads to a contradiction.

Therefore x and y have degree 4. By Lemma 38, there is an other 5-cycle containing uvw , and as G has girth at least 5, there are x' and y' distinct from all the vertices defined previously such that $uvw x' y'$ is a cycle. By symmetry, x' and y' are 4-vertices. Let $H^* = G - \{u, v, w, x, y, x'\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^* . Adding u, v and w to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction.

Therefore by Lemmas 36, 37, and 39, every 2-vertex is only adjacent to 2-vertices, so either G does not have any 2-vertex, or it is 2-regular. If G is 2-regular, then G is a n -cycle and thus $m = n$. Since $G \in \mathcal{P}_5$, we have $n \geq 5$. It is clear that G has an induced forest of size $n - 1$. Recall that $a \leq 5b$ and $a \leq 1$; this gives that $5(a - b) \leq 4$. Since $n \geq 5$, we can deduce that $an - bm = (a - b)n \leq n - 1$. This contradicts the fact that G is a counter-example. Therefore, G has minimum degree at least 3. This completes the proof. \square

Lemma 43. *Let $v_0 v_1 v_2 v_3 v_4$ be a 5-cycle in G such that v_0 is a 4-vertex and the other v_i are 3-vertices. The third neighbors of v_1 and v_2 are 3-vertices.*

Proof. Let $v_0 v_1 v_2 v_3 v_4$ be a 5-cycle in G such that v_0 is a 4-vertex and the other v_i are 3-vertices. Let u_i be the third neighbor of v_i for $i \in \{1, 2, 3, 4\}$. Suppose u_1 or u_2 , say u_{i_0} , is a 4-vertex. Let $H^* = G - C - u_{i_0}$. Graph H^* has $n - 6$ vertices and $m' = m - 14$ edges. Adding v_1, v_2 and v_4 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ completes the proof. \square

Lemma 44. *There is no separating 5-cycles with at most one 4-vertex in G .*

Proof. Let $C = v_0 v_1 v_2 v_3 v_4$ be such a cycle. By Lemma 41, C has exactly one 4-vertex, say v_0 . Let u_i be the third neighbor of v_i for $i \in \{1, 2, 3, 4\}$. By the girth assumption, all the u_i are distinct. By Lemma 43, all the u_i have degree 3.

Suppose C separates u_1 and u_2 . Let $H^* = G - C$. Graph H^* has $n - 5$ vertices and $m' \leq m - 10$ edges, and adding v_1, v_2 and v_4 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (5, 10, 3)$ leads to a contradiction.

So C does not separate u_1 and u_2 , and by symmetry it does not separate u_3 and u_4 either.

Suppose C separates some of the u_i . Say u_1 and u_2 are in the interior of C w.l.o.g., and u_3 and u_4 are in the exterior of C . By Lemma 38 there is a

vertex w such that $u_1v_1v_2u_2w$ is a cycle. Since u_1, v_1, v_2 and u_2 have degree 3, and v_0 has degree 4, w has degree 3 by Lemma 43. Vertex w cannot be adjacent to v_0, v_1 or v_2 by the girth assumption, and it cannot be adjacent to v_3, v_4, u_3 or u_4 by planarity. Let w' be the third neighbor of u_1 . It is also non-adjacent to all the vertices defined previously (except for u_1) by the girth assumption and planarity of G . Let $H^* = G - C - \{u_1, u_2, u_3, w, w'\}$. Graph H^* has $n - 10$ vertices and $m' \leq m - 20$ edges, and adding v_1, v_2, v_3, v_4, u_1 and u_2 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (10, 20, 6)$ leads to a contradiction.

Therefore C does not separate any of the u_i , say the u_i are in the exterior of C up to changing the plane embedding. Then as G is 2-edge-connected by Lemma 36, the two neighbors of v_0 distinct from v_1 and v_4 are in the interior of C . By Lemma 38, either $u_1u_3 \in E$, or there is a vertex w such that $u_1v_1v_2u_2w$ is a cycle. If $u_1u_3 \in E$, then the cycle $v_1v_2v_3u_3u_1$ is separating with only 3-vertices, contradicting Lemma 41. Thus $u_1u_3 \notin E$ (and $u_2u_4 \notin E$ by symmetry), and there is a vertex w such that $u_1v_1v_2u_2w$ is a cycle. Since u_1, v_1, v_2 and u_2 have degree 3, and v_0 has degree 4, by Lemma 43 w has degree 3. If $w = u_4$, then $u_2u_4 \in E$, which is impossible; hence w is not adjacent to v_4 . It is not adjacent to the other v_i by girth assumption. Let $H^* = G - \{v_1, v_2, v_3, v_4, u_1, u_2, w\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^* . Adding u_1, u_2, v_2 , and v_4 to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 14, 4)$ completes the proof. \square

Lemma 45. *Let $C = v_0v_1v_2v_3v_4$ be a 5-cycle in G with only 3-vertices, and u_i be the third neighbor of v_i for $i \in \{0, 1, 2, 3, 4\}$. Then there is a vertex w adjacent either to u_0 and u_1 or to u_2 and u_3 .*

Proof. Let $C = v_0v_1v_2v_3v_4$ be a 5-cycle with only vertices of degree 3 in G , and let u_i be the third neighbor of v_i for $i \in \{0, 1, 2, 3, 4\}$. See Figure 15 for an illustration of the statement of the lemma. By Lemma 41, C is the boundary of a face.

Let us first show that no two u_i can be adjacent. Suppose two of the u_i are adjacent. By the girth assumption, w.l.o.g. $u_0u_2 \in E$. Then by Lemma 44, u_0 and u_2 have degree 4. Let $H^* = G - C - \{u_0, u_2\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^* . Adding v_0, v_1, v_2 and v_3 to F' leads to an induced forest of G by planarity. Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 14, 4)$ leads to a contradiction.

Suppose by contradiction that there is no vertex w adjacent either to u_0 and u_1 , or to u_2 and u_3 . Let $H^* = G - C + \{x, y\} + \{u_0x, u_1x, u_2y, u_3y, xy\}$. Graph H^* is of girth at least 5 by hypothesis and because the u_i are not

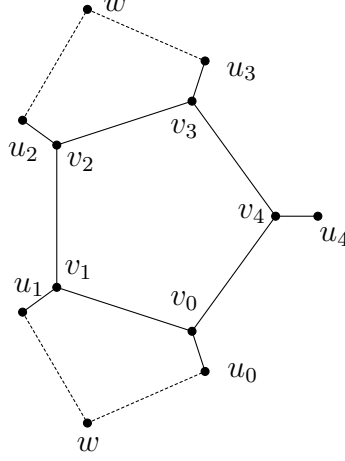


Figure 13: The construction of Lemma 45. At least one of the two w represented exists.

adjacent. Graph H^* has $n - 3$ vertices and $m' = m - 5$ edges. Let F' be any induced forest of H^* . Removing x and y , adding v_0 and v_3 , plus v_1 if $x \in F'$, and v_2 if $y \in F'$ to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$ completes the proof. \square

Lemma 46. *There is no 5-face with exactly one 4-vertex in G .*

Proof. Let $C = v_0v_1v_2v_3v_4$ be such a face, with v_0 the 4-vertex, and let u_i be the third neighbor of v_i for $i \in \{1, 2, 3, 4\}$. By Lemma 43, the u_i have degree 3. The u_i are all distinct and not adjacent to v_0 by the girth assumption. By Lemma 38, either $u_1u_3 \in E$, or there is a vertex adjacent to both u_1 and u_2 . However in the former case, the cycle $u_1v_1v_2v_3u_3$ is a separating cycle with five vertices of degree 3, contradicting Lemma 41. Hence $u_1u_3 \notin E$ and $u_2u_4 \notin E$ by symmetry. We also have $u_1u_4 \notin E$ by Lemma 44 applied to $u_1v_1v_0v_4u_4$. Let w be the vertex adjacent to both u_1 and u_2 . By Lemma 43, w has degree 3. By the girth assumption, $wv_0 \notin E$ and $wv_3 \notin E$. By Lemma 41, $v_1v_2u_2wu_1$ is the boundary of a face. Moreover, $wv_4 \notin E$ and $wu_3 \notin E$ by applying Lemma 44 to the cycle $wv_4v_3v_2u_2$ and $wu_3v_3v_2u_2$ respectively. By symmetry, let w' ($\neq w$) be the vertex adjacent to u_3 and u_4 . Vertex w has degree 3, $w'v_0 \notin E$, $w'v_1 \notin E$, $w'v_2 \notin E$, $w'u_2 \notin E$ and $u_4v_4v_3u_3w'$ is the boundary of a face.

Observe now that $wu_4 \notin E$ and $w'u_1 \notin E$ (by symmetry). By contradiction assume $wu_4 \in E$. Consider $H^* = G - \{v_0, v_1, v_2, v_4, u_1, u_4, w\}$ which has

$n - 7$ vertices and $m' \leq m - 14$ edges. Adding the vertices w, u_1, v_1 and v_4 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 14, 4)$ completes the proof.

Observe now that $ww' \notin E$. Otherwise, consider $H^* = G - \{v_0, v_1, v_4, u_1, u_4, w, w'\}$ which has $n - 7$ vertices and $m' \leq m - 14$ edges. Adding the vertices u_1, v_1, v_4 and w' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 14, 4)$ completes the proof.

See Figure 14 for a summary of the edges between the vertices $v_0, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, w$ and w' .

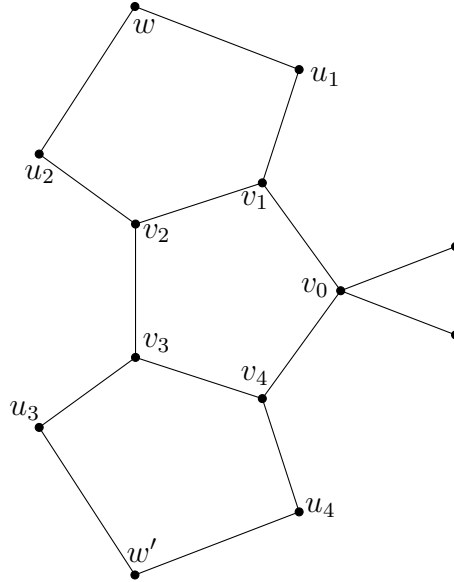


Figure 14: The vertices $v_0, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, w$ and w' , and the edges between these vertices. All the vertices except for v_0 are 3-vertices.

Let x be the third neighbor of u_1 (x is distinct from all previously defined vertices). By the girth assumption $xw \notin E$, $xu_2 \notin E$ and $xv_0 \notin E$.

Observe that $xu_4 \notin E$ and $xw' \notin E$. Otherwise consider $H^* = G - \{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, w, w', x\}$, which has $n - 11$ vertices and $m' \leq m - 19$ edges. Adding the vertices $v_1, v_2, v_3, u_1, u_4, w$ and w' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (11, 19, 7)$ completes the proof.

Similarly, $xu_3 \notin E$ (just add u_3 to F' instead of w').

Finally, let $H^* = G - C - \{u_1, u_2, u_3, u_4, w, w', x\}$. Graph H^* has $n - 12$ vertices and $m' \leq m - 23$ edges. Adding $v_1, v_2, v_3, v_4, u_1, w$ and w' to any

induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (12, 23, 7)$ completes the proof. \square

Lemma 47. *There is no 5-face $v_0v_1v_2v_3v_4$ in G such that all the v_i are 3-vertices, and three of the v_i have a 4-vertex as their third neighbor.*

Proof. Let $C = v_0v_1v_2v_3v_4$ be such a face, and let u_i be the third neighbor of v_i for $i \in \{0, 1, 2, 3, 4\}$.

Suppose two of the u_i are adjacent. By the girth assumption the corresponding v_i are not adjacent. W.l.o.g., say u_0 and u_2 are adjacent. Then since C is a face, $v_0v_1v_2u_2u_0$ is separating, and thus by Lemma 44, u_0 and u_2 have degree 4. Let $H^* = G - \{v_0, v_1, v_2, v_3, u_0, u_2\}$. Graph H^* has $n - 6$ vertices and $m' \leq m - 14$ edges. Adding v_0, v_1 and v_2 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$ leads to a contradiction. Therefore no two u_i are adjacent.

Let H^* obtained from G where we remove C and three u_i of degree 4. Graph H^* has $n - 8$ vertices and $m' \leq m - 19$ edges. Let F' be any induced forest of H^* . Adding the three v_i that correspond to the u_i we removed, plus another v_i to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$ completes the proof. \square

Lemma 48. *If there are two 5-cycles $C = v_0v_1v_2v_3v_4$ and $C' = v_0v_1u_2u_3u_4$ sharing an edge v_0v_1 in G with only 3-vertices, then for all $x \in \{u_2, u_3, u_4\}$, $xv_3 \notin E$. Moreover, for all $x \in \{u_2, u_3, u_4\}$, x and v_3 do not share a common neighbor.*

Proof. Let $C = v_0v_1v_2v_3v_4$, $C' = v_0v_1u_2u_3u_4$, and $x \in \{u_2, u_3, u_4\}$. Cycles C and C' are the boundaries of faces by Lemma 44. If x is either u_2 or u_4 , then we can conclude by the girth assumption and Lemma 44.

Consider now the case $x = u_3$. By Lemma 41, $v_3u_3 \notin E$. Finally assume that there is a vertex w adjacent to both v_3 and u_3 . Let $H^* = G - (C \cup C') - w$. Graph H^* has $n - 9$ vertices and $m' \leq m - 15$ edges. Adding v_0, v_1, v_2, v_3, u_3 and u_4 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (9, 15, 6)$ completes the proof. \square

Lemma 49. *There is no 5-face in G with only 3-vertices.*

Proof. Let $C = v_0v_1v_2v_3v_4$ be such a face, and let u_i be the third neighbors of v_i for $i \in \{0, 1, 2, 3, 4\}$. By Lemma 47, no more than two of the u_i are 4-vertices.

By the girth assumption, all the u_i are distinct and two u_i whose corresponding v_i are adjacent are not adjacent.

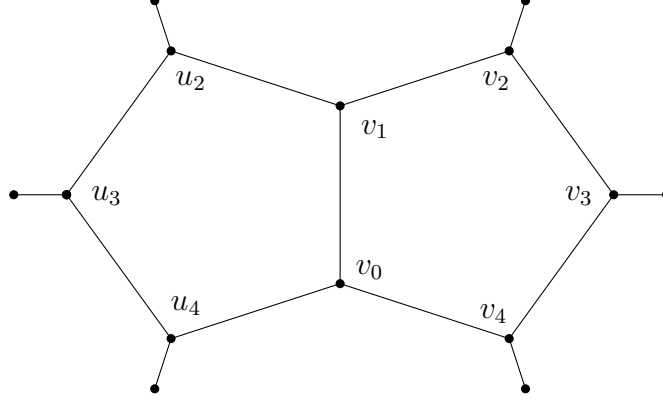


Figure 15: The construction of Lemma 48. All the edges between the vertices $v_0, v_1, v_2, v_3, v_4, u_2, u_3$ and u_4 are represented.

We prove now that there is no edge between the u_i . W.l.o.g. suppose $u_0 u_2 \in E$. By Lemma 44, u_0 and u_2 are 4-vertices. Let $H^* = G - C - \{u_0, u_2\}$. Graph H^* has $n - 7$ vertices and $m' \leq m - 14$ edges. Let F' be any induced forest of H^* . Adding v_0, v_1, v_2 and v_3 to F' leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (7, 14, 4)$ leads to a contradiction.

We now consider four cases:

- Suppose two u_i have degree 4, and the corresponding v_i are adjacent. W.l.o.g. u_0 and u_1 have degree 4.

Let us first assume that there is a vertex w adjacent to u_2 and u_3 . Vertex w has degree 3 by Lemmas 44 and 46 (in particular $w \neq u_0$). Vertex w is not adjacent to any of the v_i or u_i except for u_2 and u_3 by Lemma 48. Let $H^* = G - C - \{u_0, u_1, u_2, u_3, u_4, w\}$. Graph H^* has $n - 11$ vertices and $m' \leq m - 23$ edges. Adding v_0, v_1, v_2, v_4, u_2 and u_3 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (11, 23, 6)$ leads to a contradiction.

So there is no vertex w adjacent to u_2 and u_3 , and by symmetry there is no vertex w adjacent to u_3 and u_4 . By Lemma 45 there is a vertex w' adjacent to u_4 and u_0 . By Lemmas 44 and 46, w' has degree 4. By Lemma 38, since there is no edge among the u_i and by the girth assumption, there is a vertex w adjacent to u_3 and u_4 , a contradiction.

- Suppose two u_i have degree 4, and the corresponding v_i are not adja-

cent. W.l.o.g. u_0 and u_2 have degree 4. Then by Lemma 45 there is a vertex w' adjacent either to u_0 and u_4 or to u_2 and u_3 . W.l.o.g. w' is adjacent to u_2 and u_3 . By Lemmas 44 and 46, w' has degree 4. By Lemma 38, since there is no edge among the u_i and by the girth assumption, there is a vertex w adjacent to u_3 and u_4 . Vertex w has degree 3 by Lemmas 44 and 46. Vertex w is not adjacent to any of the v_i or u_i except u_3 and u_4 by Lemma 48. Let $H^* = G - C - \{u_0, u_1, u_2, u_3, u_4, w\}$. Graph H^* has $n - 11$ vertices and $m' \leq m - 23$ edges. Adding v_0, v_1, v_2, v_4, u_3 and u_4 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (11, 23, 6)$ leads to a contradiction.

- Suppose exactly one u_i has degree 4, u_0 w.l.o.g., and u_0 is adjacent to a vertex w that is adjacent to either u_1 or u_4 , say u_1 . Vertex w has degree 4 by Lemmas 44 and 46. By Lemma 38, since there is no edge among the u_i and by the girth assumption, there is a vertex w' adjacent to u_1 and u_2 . Moreover w' has degree 3 by Lemmas 44 and 46. Vertex w' is not adjacent to any of the v_i or u_i except for u_1 and u_2 by Lemma 48. By Lemma 45, there is a vertex w'' adjacent either to u_2 and u_3 or to u_0 and u_4 .

Suppose w'' is adjacent to u_2 and u_3 . By Lemmas 44 and 46, w'' has degree 3, and w'' is not adjacent to any of the v_i or u_i except u_2 and u_3 by Lemma 48. By the girth assumption, $w'w'' \notin E$ and $ww' \notin E$. By Lemmas 44 and 46, $ww'' \notin E$. By Lemma 48 applied to $v_2u_2w''u_3v_3$ and $v_1u_1w'u_2v_2$, $wu_3 \notin E$. Let $H^* = G - C - \{u_0, u_1, u_2, u_3, w, w', w''\}$. Graph H^* has $n - 12$ vertices and $m' \leq m - 23$ edges. Adding $v_0, v_2, v_4, u_1, u_2, u_3$, and w' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (12, 23, 7)$ leads to a contradiction.

Thus w'' is adjacent to u_0 and u_4 . By the same arguments as above, w'' being the symmetrical of w , w'' has degree 4 and there is a 3-vertex w''' adjacent to u_3 and u_4 , and not to any other of the u_i and v_i .

Suppose $w'w''' \in E$. Let $H^* = G - C - \{u_1, u_2, u_3, u_4, w', w'''\}$. Graph H^* has $n - 11$ vertices and $m' \leq m - 19$ edges. Adding $v_1, v_2, v_3, v_4, u_3, u_4$ and w' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (11, 19, 7)$ leads to a contradiction.

Thus $w'w''' \notin E$. Recall that w' and w''' are not adjacent to any of the v_i or u_i except for u_1 and u_2 , and u_3 and u_4 respectively. Let $H^* = G - C - \{u_0, u_1, u_2, u_3, u_4, w', w'''\}$. Graph H^* has $n - 12$ vertices

and $m' \leq m - 23$ edges. Adding $v_0, v_1, v_2, v_3, u_3, u_4$ and w' to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (12, 23, 7)$ leads to a contradiction.

- Thus either all the u_i have degree 3, or u_0 has degree 4 and there is no w adjacent to u_0 and either to u_1 or to u_4 . In both cases u_1, u_2, u_3 and u_4 have degree 3, and, w.l.o.g., by Lemma 45 there are vertices w_1, w_2 and w_3 adjacent to u_1 and u_2 , to u_2 and u_3 and to u_3 and u_4 respectively. For all $j \in \{1, 2, 3\}$, by Lemmas 44 and 46, w_j has degree 3, and by Lemma 48, w_j is not adjacent to any of the u_i and v_i except for u_j and u_{j+1} . We have $w_1w_2 \notin E$ and $w_2w_3 \notin E$ by the girth assumption, and $w_1w_3 \notin E$ by Lemma 41. Let $H^* = G - C - \{u_0, u_1, u_2, u_3, u_4, w_1, w_2, w_3\}$. Graph H^* has $n - 13$ vertices and $m' \leq m - 23$ edges. Adding $v_0, v_1, v_2, v_3, u_1, u_2, u_3$ and u_4 to any induced forest of H^* leads to an induced forest of G . Observation 35 applied to $(\alpha, \beta, \gamma) = (13, 23, 8)$ completes the proof.

□

Each 4-vertex is in the boundary of at most four faces. Therefore the sum of the $c_4(f)$ over all the 5-faces is $\sum_{f, l(f)=5} c_4(f) \leq 4n_4$. From Lemmas 46 and 49 we can deduce that for each 5-face f we have $c_4(f) \geq 2$. Thus $\sum_{f, l(f)=5} c_4(f) \geq 2k_5$. Thus we have the following:

$$4n_4 \geq 2k_5$$

By Euler's formula, we have:

$$\begin{aligned} -12 &= 6m - 6n - 6k \\ &= 2 \sum_{v \in V(G)} d(v) + \sum_{f \in F(G)} l(f) - 6n - 6k \\ &= \sum_{d \geq 3} (2d - 6)n_d + \sum_{l \geq 5} (l - 6)k_l \\ &\geq 2n_4 - k_5 \\ &\geq 0 \end{aligned}$$

This is a contradiction, which ends the proof of Theorem 34.

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